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ABSTRACT

Research recurrently indicates that children who have difficulty with arithmetic often use systematic routines that yield wrong answers. Recent research has focused less on identifying the most common errors among groups of children and more on analyzing individual children's errors. This paper considers the source of systematic errors in subtraction with multidigit numbers and appropriate instructional responses. The author argues that conceptual misunderstandings lie at the heart of many errorful procedures, and that these misunderstandings are what should be addressed in instruction, as well as the problem of linking conceptual understanding to procedural skill. Following an analysis of the nature of "buggy algorithms" in subtraction, teaching the semantics of procedures is considered. Principles of subtraction and place value are presented, and children's knowledge of the principles is reviewed. Instructional experiments on how the principles apply to written arithmetic are described. Finally, conclusions and some questions are presented. The importance of error analysis research is confirmed. Systematic errors probably arise from a basic failure to mentally represent arithmetic procedures in terms of operations on quantities within a principled number system, rather than as operations on symbols that obey largely syntactic rules. (MNS)

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Beyond Error Analysis: The Role of Understanding in Elementary School Arithmetic

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One of the recurrent findings in research on children's arithmetic performance is that children who are having difficulty with arithmetic often use systematic routines that yield wrong answers. This observation has been made repeatedly, and a number of studies have attempted to describe the most common errors. Buswell (1927) used a combination of eye movement and primitive solution-time measures, pencil and paper tests constructed for diagnostic purposes, and what we would now call "thinking-aloud protocols" to determine the most common sources of errors in calculation with whole numbers. Brownell (1928, 1935) conducted similar studies in his many years of work on the psychology of mathematics education.

These investigators were seeking to put into the hands of teachers tools that would help them discover the basis of individual pupils' arithmetic difficulties, so that appropriate remedial instruction could be offered. The work of Buswell and Brownell and of other early psychologists and mathematics educators was clearly guided by a recognition that children's errors were systematic. Their research, however, did not attempt to describe individual children's calculation strategies but concentrated instead on identifying the most common errors among groups of children.

Recent research has focused more on the analysis of individual children's errors. Lankford (1972), for example, using a diagnostic interview procedure with seventh graders, made it clear that the students' computational strategies were highly individual, often not following the orthodox models of textbook and classroom. Some of the unorthodox strategies were successful, others were not. Among mathematics educators, members of the Research Council on Diagnostic and Prescriptive Mathematics have often taken the lead in uncovering regularities in children's errorful arithmetic, with an eye to adapting instruction to individual error patterns. Among cognitive scientists, a similar line of investigation on "buggy algorithms" (Brown & Burton, 1978) has not only documented the existence of consistent error-producing algorithms, but has developed automated diagnostic programs capable of reliably detecting the particular errorful algorithms used by a child on the basis of responses to a very small but carefully selected number of problems.

The phenomenon of systematic errors in calculation then, is well established. What is less certain is (a) the source of these errorful but systematic "inventions" by children; and (b) the appropriate instructional response to the observation that errors are systematic. These are the questions I address in this paper. I use the domain of subtraction of multi-digit numbers as my primary example, in order to use the extensive theoretical analyses and experimental investigations of errors in this domain that are available to me. My concern is not for subtraction as such, however, but for certain more general principles and questions that the case of subtraction illustrates.

Let me state at the outset that I intend to cast doubt on the proposition that error analyses themselves, no matter how detailed, will yield the kind of diagnoses that provide a strong basis for instruction. Instead, I argue that for any given domain of arithmetic a relatively small number of conceptual misunderstandings lie at the heart of many different errorful procedures. It is these conceptual misunderstandings that should be addressed in instruction. I will also show that conceptual understanding does not always automatically produce correct procedures. Thus, instruction also must address directly the problem of linking conceptual understanding to procedural skill—a task that seems likely to prove more complex than many of us once believed.

The Nature of Buggy Algorithms in Subtraction

Let us begin by establishing the nature of the problem in our example domain of multi-digit subtraction. Figure 1 shows some of the most common subtraction errors that have been identified by Brown and his colleagues in their extensive work on buggy algorithms in subtraction. This is only a partial list of the known and demonstrated subtraction bugs, but it is enough to allow us to consider the possible sources of buggy procedures in this domain of arithmetic.

Two theories of the origin of subtraction bugs have been proposed. One, by Young and O'Shea (1981), suggests that the simpler bugs arise when children either forget or have never learned the standard school-taught subtraction algorithm (see Figure 2 for this algorithm). The second, by Brown and VanLehn (1980, 1982), is known as "repair theory." According to repair theory, buggy algorithms arise when an arithmetic problem is encountered for which the child's current algorithms are incomplete or inappropriate. The child, trying to respond, eventually reaches an impasse, a situation for which no action is available. At this point, the child calls on a list of "repairs"—actions to try when the standard action cannot be used. The repair list includes strategies such as performing the action in a different column, skipping the action, swapping top and bottom numbers in a column, and substituting an operation (such as incrementing or decrementing). The outcomes generated through this repair process are then checked by a set of "critics" that inspect the resulting solution for conformity to some basic criteria, such as no empty columns, only one digit per column in the answer, only one decrement per column, and the like. Together, the repair and the critic lists constitute the key elements in a "generate and test" problem-solving routine.

This model's operations demonstrate the same kind of "intelligent" problem solving that characterizes

1. Smaller-From-Larger. The student subtracts the smaller digit in a column from the larger digit regardless of which one is on top.

326 − 117 = 211   542 − 389 = 247

2. Borrow-From-Zero. When borrowing from a column whose top digit is 0, the student writes 9 but does not continue borrowing from the column to the left of the 0.

642 − 437 = 265   802 − 396 = 506

3. Borrow-Across-Zero. When the student needs to borrow from a column whose top digit is 0 he skips that column and borrows from the next one. (Note: this bug requires a special "rule" for subtracting from 0: either 0 - N = N or 0 - N = 0.)

602 − 327 = 335   704 − 456 = 308

4. Stop-Borrow-At-Zero. The student fails to decrement 0, although he adds 10 correctly to the top digit of the active column. (Note: this bug must be combined with either 0 - N = N or 0 - N = 0.)

703 − 678 = 175   604 − 387 = 307

5. Don't-Decrement-Zero. When borrowing from a column in which the top digit is 0, the student rewrites the 0 as 10 but does not change the 10 to 9 when incrementing the active column.

702 − 368 = 344   805 − 9 = 1106

6. Zero-Instead-Of-Borrow. The student writes 0 as the answer in any column in which the bottom digit is larger than the top.

326 − 117 = 210   542 − 389 = 200

7. Borrow-From-Bottom-Instead-Of-Zero. If the top digit in the column being borrowed from is 0, the the student borrows from the bottom digit instead. (Note: this bug must be combined with either 0 - N = N or 0 - N = 0.)

702 − 389 = 454   508 − 489 = 109

Figure 1. Common subtraction errors identified by J.S. Brown, R.R. Burton, and K. VanLehn. (From Resnick, 1982.)

many successful performances in other domains (cf. Simon, 1976, pp. 65, 98). With buggy algorithms, the trouble seems to lie not in the reasoning processes but in the inadequate data base applied. Inspection of the repair and critic lists makes it clear that the generation and the test rules in this particular system can all be viewed as "syntactic." That is, they all concern the surface structure of the subtraction procedure and do not necessarily reflect what we can call the "semantics," or underlying meaning, of the procedure.

This distinction between the syntax and the semantics of the procedure becomes clearer when we consider some of the individual bugs. Inspection of the bugs in Figure 1 shows that they tend to "look right" and to obey most of the important syntactic rules for written calculation: The digit structure is respected, there is only a single digit per column, all the columns are filled, and so forth. In the sense of being an orderly and reasonable response to a problem situation, the buggy algorithms look quite sensible. But each of the bugs violates fundamental mathematical constraints. In this sense, they violate the conceptual meaning, or semantics, of subtraction. I can make this point more clearly by individually considering each of the bugs in Figure 1.

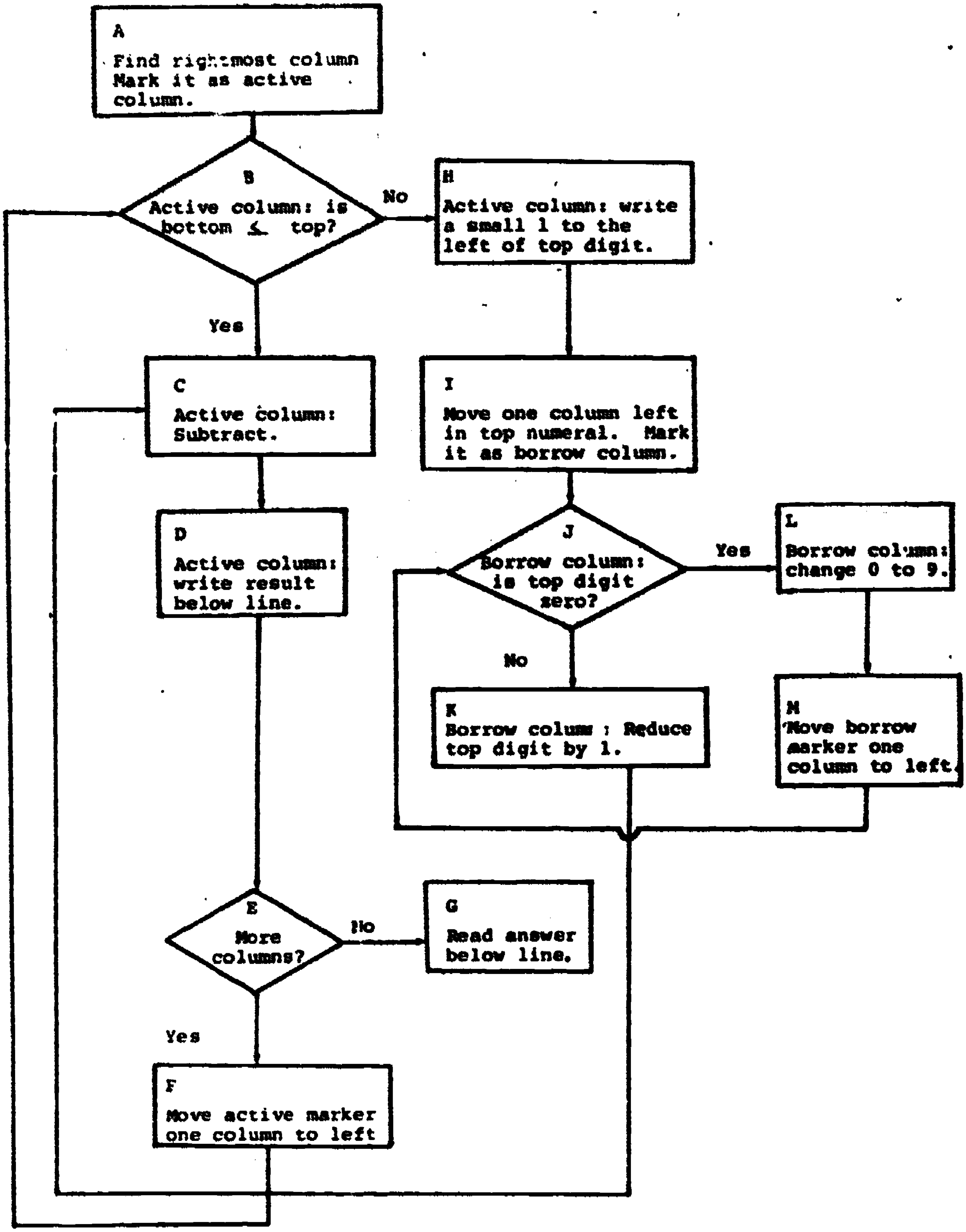

Figure 2. Standard algorithm for subtraction. (From Resnick, 1982.)

1. Smaller-From-Larger. Repair theory suggests that this very common bug results from "switching arguments" to respond to a situation in which the child cannot make the normal move of subtracting the bottom from the top number in a column. In other words, the child makes the test at B in Figure 2 but does not know how to borrow and decides that the subtraction should be done in the opposite direction. Young and O'Shea's analysis suggests that this bug derives from simply not making the test and is the normal, or default, way for a child to proceed unless the test is made and the various borrowing rules are thereby evoked. In both of these interpretations, all the syntax of written subtraction without borrowing is respected. What is violated is the constraint that the bottom quantity as a whole be subtracted from the top quantity as a whole. The semantics of multi-digit subtraction include the constraint that the columns, although handled one at a time, cannot be treated as if they were a string of unrelated single-digit subtraction problems.

2. Borrow-From-Zero. Both repair theory and the Young and O'Shea analysis suggest that this bug derives from forgetting the part of the written procedure that is equivalent to steps M-J-K in Figure 2 (moving the borrow marker left, and reducing the new column). The bug respects the syntactic requirement that, in a borrow, there must be a crossed-out and rewritten numeral to the left of the active column. It also respects the syntax of the special case of zero, where the rewritten number is always 9. However, it ignores the fact that the 9 really results from borrowing one column further left (the hundreds column) moving 100 as 10 tens into the tens column, and then borrowing from the 10 tens leaving 9 tens, or 90 (written as 9).

3. Borrow-Across-Zero. Repair theory offers two different derivations of this bug. The first is that this bug arises from the child's search for a place to do the decrementing operation with the condition that the column not have a zero in the top number. This would happen when the child doesn't know how to handle zeros or thinks they have "no value" and thus can be skipped. This solution respects the syntactic constraint that a small 1 must be written in the active column and that some other (nonzero) column must then be decremented. But the semantic knowledge that the increment and decrement are actually addition and subtraction of 10 is ignored (or not known). Repair theory's second derivation, which agrees with Young and O'Shea's analysis, produces this bug by simply deleting the rule that changes 0 to 9 (L in Figure 2). This too is a completely syntactic derivation, for it allows deletion of a rule without reference to the semantic information that justifies the operation.

4. Stop-Borrow-At-Zero. Both repair theory and Young and O'Shea's analysis interpret this bug as simply omitting a rule or an operation. Steps I-J-K of Figure 2 are simply skipped. This bug fails to obey both syntactic and semantic constraints. Syntactically it produces only the increment part of the borrow operation--the 1 in the active column--but does not show a crossed-out number or the change of a 0 to a 9. Semantically, it violates the justification for the borrow increment--that is, in order to add a quantity to the active column an equivalent quantity must be subtracted from another column.

5. Don't-Decrement-Zero. The change of 0 to 10 in this bug is the proper "semantic" move after borrowing from the hundreds column. But it produces an outcome that the child may not have encountered and thus does not respond to appropriately. Failure to change the 10 to 9 may result from a syntactic constraint that each column be operated on only once. This syntactic constraint is not "correct" but might be reasonably inferred from extensive experience with problems that contain no zeros. If so, the syntactic constraint is in direct opposition to the semantic demands of the situation.

6. Zero-Instead-Of-Borrow. Like Smaller-From-Larger, this bug avoids the borrowing operation altogether, while observing all of the important syntactic constraints of operating within columns, writing only one small digit per column, and the like. This bug, however, does not violate the semantics of the digit structure as blatantly as the Smaller-From-Larger bug. In fact, a child producing this bug may be following a semantics of subtraction that generally precedes any understanding of negative numbers. In this inferred semantics of subtraction, when a larger number must be taken from a smaller, the decrementing is begun and continued until there are no more left--yielding zero as the answer.

7. Borrow-From-Bottom-Instead-Of-Zero. This bug appears to be purely syntactic in the sense that the search for something to decrement appears to lead the child to ignore the digit structure and the semantics of exchange that justify borrowing within the top number. But it does produce a "funny-looking" solution, so it would probably be generated only by a child whose syntactic rules did not specifically require that all increments and decrements be in the top number.

Teaching the Semantics of Procedures

The analysis just presented suggests that children who invent buggy algorithms are already paying attention to the surface aspects of the calculation routines they are taught. Their invented algorithms deviate only slightly from the correct ones, and by and large they respect the strictly procedural rules of written subtraction. The trouble seems to lie in keeping details of procedural rules (e.g., when and where to decrement) in mind and combining the various rules correctly. Two instructional responses to this procedural confusion can be imagined. The first is to focus directly on the procedural rules, designing practice and feedback that will focus children's attention on the specific points at which they misapply the rules. Here it would be ideal to adapt the instruction to individual children's errors, although "standardized" instruction to address the most common buggy algorithms could also be designed. The second response is to focus instruction on the semantic principles that are violated in invented buggy algorithms, in the hope that children will then bring these principles into play when they face subtraction problems for which their learned rules are incomplete.

Principles of Subtraction and Place Value

One of the reasons for preferring semantically based instruction over purely procedural instruction is its potential economy. There are many specific procedural rules to learn and many specific problems to apply them to, all of which would need to be addressed in rule-based instruction. There are also a number of different mathematically correct algorithms for subtraction. Rule-based instruction would either have to limit "acceptable" responses to just one of these algorithms or systematically teach several of them (a costly matter, not only because it would take a great deal of time but because it would probably induce confusions between algorithms and thus produce some new bugs). All of the rules, however, and all of the different algorithms of subtraction are expressions of a very few basic principles. These are:

1. The difference principle. The goal of subtraction is to find the difference between two quantities (which we call "bottom" and "top" for convenience here).

2. The composition principle. Each of these two quantities may be expressed as compositions of smaller quantities (e.g., hundreds, tens, units). What is sought, however, is the difference between the bottom quantity as a whole and the top quantity as a whole.

3. The partition principle. It is permissible to find the difference by partitioning the quantities into any convenient subquantities (parts), finding the difference between each bottom part and one of the top parts, and then recomposing the results found into a single quantity. The partition principle is what permits column by column written subtraction. It also permits a variety of mental subtraction procedures, some of which are described below.

4. The compensation principle. It may be convenient when subtracting by partitioning to increment or decrement the parts in one of the quantities. This is permissible as long as an increment is compensated by a decrement (or vice versa) such that the value of the total quantity is preserved. The compensation principle is what permits borrowing.*

5. The value principle. The specific increments and decrements that are permissible to satisfy the compensation principle depend on the values of each digit in the written numbers. These values are determined by the column in which a digit is written: The value of a digit is the digit times the column value.

Children's Knowledge of the Principles

Several bodies of research on addition and subtraction knowledge suggest that most children of 9 or 10 years of age, including those who use buggy subtraction algorithms, already know a great deal about the five principles outlined above. This knowledge is evident when they work in number representations other than the written--such as money, the abacus, or Dienes blocks--or when they do mental arithmetic. Some examples of performance that give evidence of knowledge of these principles are the following:

Oral counting. There is some evidence that the compositional structure of numbers arises first in the context of oral counting. Several investigators (Fuson, Richards, & Briars, 1982; Siegler & Robinson, 1982) have found that many 4- and 5-year-olds can orally count well into the decades above 20 and that their counting shows evidence of being organized around the decade structure. For example, the most common stopping points in children's counting are at a number ending in 9 or 0 (e.g., 29 or 40), and their omissions in the number string tend to be omissions of entire decades (e.g., "...27, 28, 29, 50..."). Children also sometimes repeat entire decades (e.g., "...38, 39, 20, 21...") and sometimes make up nonstandard number names reflecting a concatenation of the tens and the units counting strings (e.g., "...twenty-nine, twenty-ten, twenty-eleven..."). Finally, children can usually succeed in counting on within a decade higher than their own highest stopping point when asked by the experimenter to start counting from a particular number, such as 51 or 71. This is, I think, an important precursor of the composition principle--that is, of the idea that numbers are composed of smaller numbers.

Quantifying block and money displays. The composition principle is more directly evidenced in children's ability to quantify and compare sets of objects that are coded for decimal value (see Figure 3 for examples of such displays). These performances also display knowledge of the value principle. In our own research on place value we have found that the typical method children use in this kind of task is to begin with the largest denomination and enumerate the blocks of that denomination using the appropriate decimally structured counting string. A successful quantification of the display in Figure 3a, for example, would produce the counting string: "100, 200, 300, 400, 410, 420, 430, 440, 450, 460, 461, 462, 463." A few children, mainly those who show the most sophisticated knowledge of other aspects of place value, count all denominations by ones and then "multiply" by the appropriate value (e.g., for Figure 3a: 1, 2, 3, 4, 400; 1, 2, 3, 4, 5, 6, 460; 1, 2, 3, 463). However, counting using the decimally structured number strings seems to be the earliest application of composition and value principles to the task of quantifying sets.

Other performance characteristics of children in an early stage of decimal number knowledge suggest that children typically recognize the relative values of

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*The compensation principle is actually only one way of satisfying a more general principle, which is that the difference between the top and bottom quantities must be maintained no matter what exchanges are made between parts. One way to keep the difference constant is to change neither quantity. Another way is to change both quantities by the same amount. This is what is done in a subtraction algorithm taught in many countries.

The principles given in the body of the text are those that justify the commonly taught subtraction algorithm in United States schools and a number of variations that (a) do not involve changing the bottom number and (b) assume that only positive integers will be used throughout the procedure.

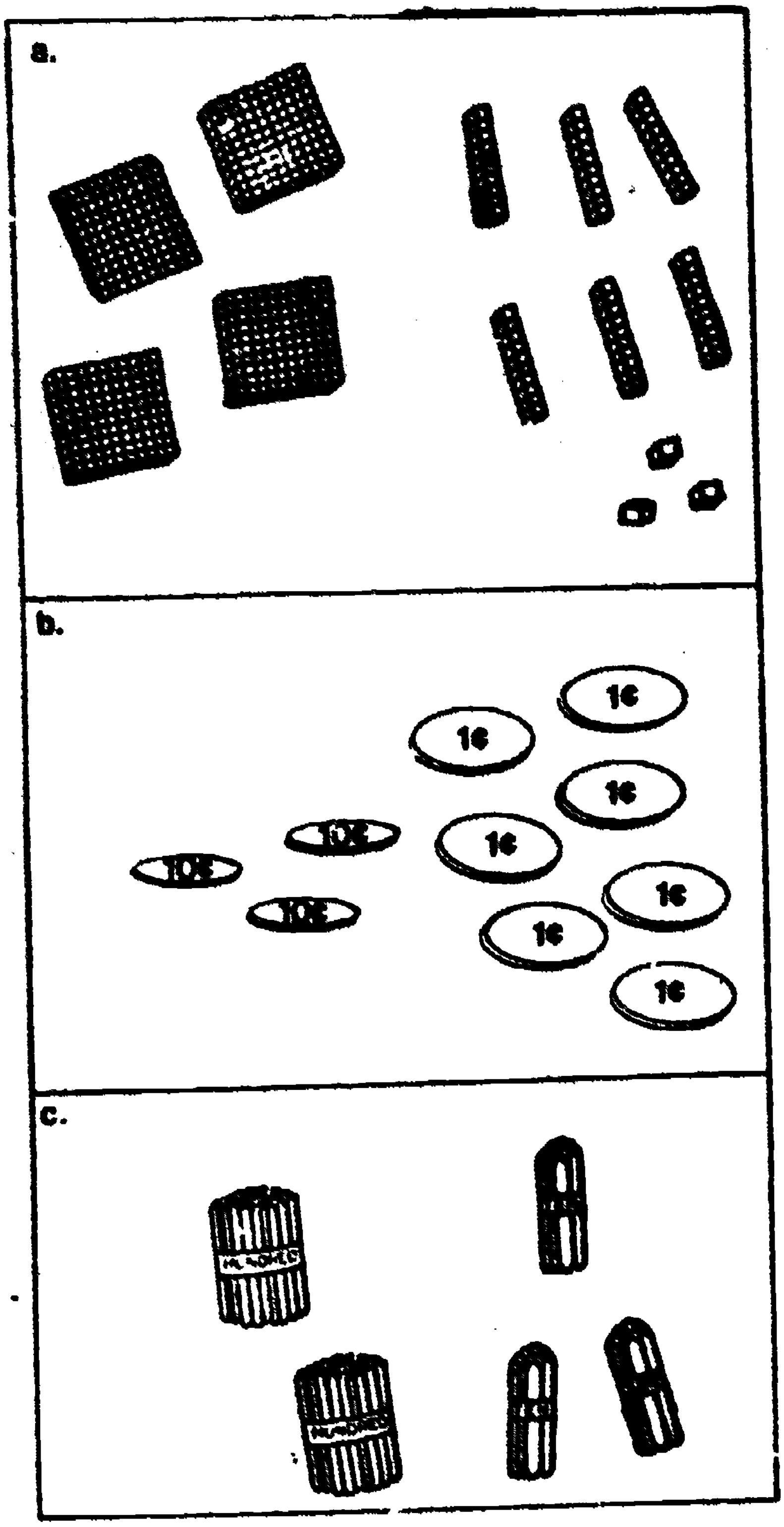

Figure [illegible]. Sets of objects coded for decimal value and used for place value tasks with children. (From Resnick, 1983.)

the different parts that make up the whole number. For example, most second through fourth graders we have interviewed compared numbers "lexicographically." That is, they first compare the highest-value digit in the two numbers and declare the number with the higher digit to be larger. Only if the two digits are the same do they go on to compare the digits to the right. For example, when comparing block displays for the numbers 472 and 427, a child would typically say 472 was larger "because it has 7 tens (or 70) and the other only has 2 tens."

<u>Mental arithmetic</u>. The most stunning diplays of the compositional and value principles are in children's invented mental calculation methods. These methods almost always involve application of the partition principle as well. Consider the following performance by one of our 8-year-old subjects, Amanda:

> E: Can you subtract 27 from 53?
>
> A: 34.
>
> E: How did you figure it out?
>
> A: Well, 50 minus 20 is 30. Then take away 3 is 27 and plus 7 is 34.

Amanda came up with the wrong answer, but by a method that clearly displayed her understanding of the compositional structure and value of two-digit numbers and of the partition principle for subtraction. She first decomposed each of the numbers in the problem into tens and units, and then performed the appropriate subtraction operation on the tens components. Next she proceeded to add in and subtract out the units components. She should have subtracted 7 and added 3, but instead reversed the digits. Amanda performed on other problems without this difficulty, yielding correct answers. Other children have shown similar strategies.

In our laboratory we have also explored decimal-based mental arithmetic using reaction time methods as children attempt to solve problems in which a single-digit number is added to a two-digit number. This is an extension of research on children's invented algorithms for single-digit addition and subtraction algorithms (e.g., Groen & Resnick, 1977; Woods, Resnick, & Groen, 1975; Svenson & Hedenborg, 1979). That research showed that young children do mental arithmetic as if they had a "counter in the head" that can be set to any number, incremented or decremented by one any number of times, and then "read out" to yield an answer. The pattern of reaction times for a set of problems depends on the number of increments or decrements required in the particular algorithm the child uses. For our two-digit plus one-digit problems there are two basic mental counting procedures, one that ignores the composition principle and another that uses that principle to reduce the number of counts needed:

1. Set the mental counter to the two-digit number, then add in the single-digit number in increments of one. Reaction times would be a function of the single-digit number (in our experimental problems, always the second number). We call this the <u>Min of the Addends</u> procedure. No use of the composition principle is made in this procedure.

2. Decompose the two-digit number into a tens component and a ones component, then recombine the tens component with whichever of the two units quantities is larger. Set the counter to this reconstituted number and then add in the smaller units digit in increments of one. For example, for 23 + 9, the counter would be set at 29 and then incremented 3 times to a sum of 32. Reaction time would be a function of the smaller of the two units digits, so the procedure is called <u>Min of the Units</u>. This procedure is

a simple version of the one Amanda used; it clearly depends on the composition principle.

We fit each of these models to the reaction times (for correct solutions only) of each of our subjects. Several children's data clearly fit the Min of the Units prediction. These children usually also described this procedure when asked in interviews how they did these problems. It is important to note that several of the children who showed evidence of these partition-based mental arithmetic strategies were among those having difficulty with written subtraction. Indeed, their buggy subtraction rules were in some cases our reason for including the children in our research samples. Thus, mastery of the value and composition principles in the context of mental arithmetic does not automatically transfer to written procedures.

Multiple representations of quantity. An early appreciation of the value and composition principles is evidenced in the ways in which children construct and "read" block and other displays, as described above. In the cases described so far, however, each quantity had only one block representation: a "canonical" representation, with no more than nine blocks per column. In this canonical display there exists a one-to-one match between the number of blocks of a particular denomination and the digit in a corresponding column in standard written notation. Insistence on the canonical form means that there is no basis for carrying and borrowing--or, in block displays, for exchanges and multiple representations of a quantity. There is no basis, in other words, for the compensation principle and thus for borrowing or carrying. A little later in their mathematical development, children show that they understand that a quantity can be represented in more than one way. This multiple representation is based on the compensation principle.

At first, children can construct alternative representations only through an empirical counting process. The following performance of one of our subjects illustrates this method. Molly was asked to use Dienes blocks to subtract 29 from 47. She began by constructing the block display that matched the larger number--that is, four tens and seven units. She then tried to remove nine units and, of course, could not. The experimenter asked if she could find any way to get more units. Molly responded by putting aside all of the units blocks and one of the tens in her display, leaving just three tens. She counted these by tens ("10, 20, 30") and then continued counting by ones, adding in a units block with each count, up to 47. On the next subtraction problem, 54 - 37, Molly began with a noncanonical display of the top number. That is, she put out four tens and counted in units blocks until she reached 54, yielding a final display of four tens and fourteen units. Molly thus appeared to have learned that certain problems will require noncanonical displays; she had incorporated into her plan for doing block subtraction a check for whether there were more units to be removed than the canonical display would provide. However, at this stage she was able to establish the equivalencies of the canonical displays only by the counting process.

Later, children become able to construct or recognize alternative displays without the need for recounting the quantity. Instead of counting, they trade blocks--for example, they discard a tens block and count in 10 units, or they discard a hundreds block and count in 10 tens. Once children regularly trade blocks rather than recounting them, they often become annoyed or amused with the experimenter who keeps asking them how they know that the display still shows the same number. They indicate the various ways that they believe that if a ten-for-one trade has been made, the total quantity could not have changed. These children clearly demonstrate understanding of the compensation principle.

Subtraction with blocks. Children's ability to perform subtraction problems using Dienes blocks or similar value-coded tokens also reveals their understanding of the five principles under consideration here. In each of our studies, even children who were having difficulty with written arithmetic typically either knew, or quickly discovered, how to subtract with blocks. A typical block subtraction procedure used by children was to display the top number canonically and then work left to right (i.e., starting with the highest-valued block denomination) to remove the number of blocks specified in the bottom number. When there were not enough blocks of some denomination, they traded with the adjacent column. This kind of procedure--especially when children show that they can use it flexibly (for example, working from right to left at the experimenter's request)--demonstrates an understanding of the difference and partition principles, as well as of the other three principles already discussed.

Applying Principles to Written Arithmetic: Instructional Experiments

We see, then, that children--even those having difficulty with written arithmetic--frequently have substantial semantic knowledge about procedures that operate on concrete representations of number. It therefore seems likely that a useful method for assisting children in the development of a semantic interpretation of written arithmetic would be to call their attention to correspondences between the steps in written arithmetic and the performance of addition and subtraction with concrete materials (cf. Dienes, 1966). One method for doing this, which we have called mapping instruction, requires the child to perform the same problem in blocks and in writing, alternating steps between the two. Under these conditions the written notations can be construed as a "record" of actions on the blocks. Figure 4 summarizes this process for a subtraction problem.

In our initial work, mapping instruction was successfully used with several children who had buggy subtraction algorithms (Resnick, 1982). Not only did their bugs disappear, but the children demonstrated that they had acquired an understanding of the semantics of the written algorithm. Figure 5 gives two examples of the kind of explanation of written borrowing that the children constructed--explanations that were not provided in the instruction. In the first example, Molly was asked to check another child's work. She knew the 10 in the tens column should be changed to 9. Her explanation in terms of the values of the decrement and increment marks (nine tens in the tens column plus one ten in the units column) clearly implies that a whole-preserving exchange must be made; otherwise, she would not have sought the "other

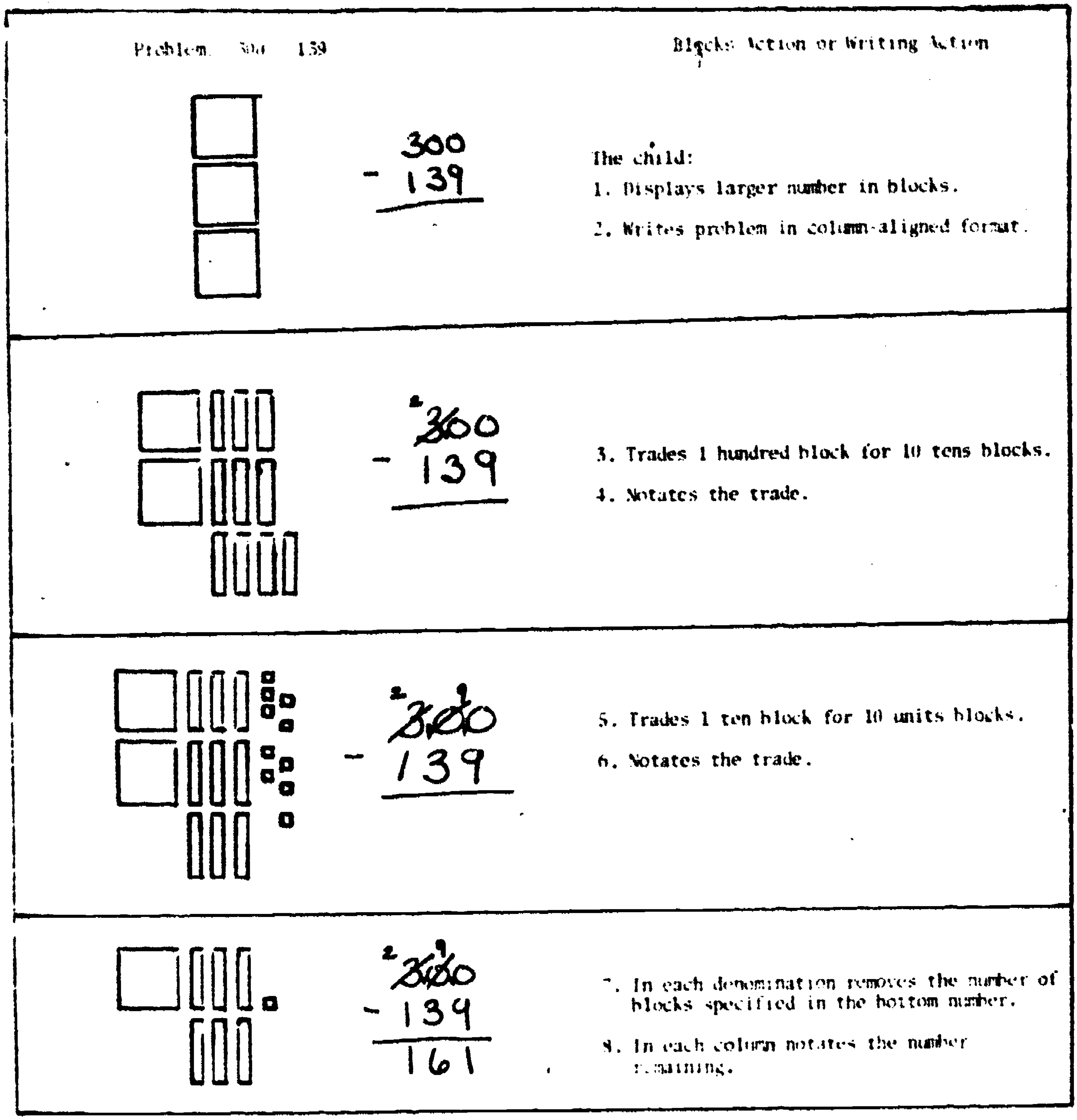

Figure 4. Summary of mapping instruction for subtraction. (From Resnick, 1982.)

ten". In the second extract, Molly shows even more clearly that she was searching for parts to make up the thousand that she recognized had been borrowed in the course of decrementing the thousands column. Both of these examples give evidence of application of the compensation principle to written borrowing.

With these encouraging results in hand, we turned next to a more formal instructional experiment, one that would not only establish the effects of mapping instruction but also would compare mapping instruction with a form of instruction that worked more directly on children's buggy algorithmic performance. The study was designed and conducted by Susan Omanson (1982). One group of children was taught by the mapping method described above and a second by a method we called prohibition instruction.

The idea in prohibition instruction was to directly prohibit incorrect steps in the written algorithm, without reference to Dienes blocks. To do this, the experimenter (E) began the prohibition instruction by introducing herself as the student's subtraction robot, who would do problems for the student (S) but who needed explicit directions about what to write. E and S worked through a set of five problems—the same five that were used in mapping instruction—with S telling E what to do. If S told E to do anything wrong or in the wrong order, E said, "I am not programmed to do it that way. Try again." Otherwise, E wrote what S told her to write. If S could not tell E the correct move after a few guesses, E wrote the correct move and then asked S to continue from there. After doing the first five problems, E and S worked through a list of about 20 more problems, each of which was designed to elicit certain bugs.

Both Mapping and Prohibition subjects were interviewed extensively before and after the instruction in order to establish their understanding of subtraction principles, as revealed in their ability to explain and justify written subtraction procedures. Their calculation performance was assessed before and after instruction, using the diagnostic test for buggy subtraction developed by Burton (1981). A delayed posttest assessed the extent to which learning was maintained over several weeks.

In one respect, the results were as we expected. The Mapping, but not the Prohibition, children improved significantly in their understanding of written subtraction. Figure 6 compares the two groups on responses that assessed understanding of the value principle; Figure 7 compares them on responses that assessed understanding of the compensation principle. Figure 8 shows the improvement, or lack of it, in individual children in the two groups. Each child's level of understanding on a 1 to 5 scale developed for this study at the pretest is shown in a circle labeled P; each child's level of understanding at the second (delayed) posttest is shown in a circle labeled p2. As can be seen, several Mapping children progressed through one or more levels of understanding, but only one Prohibition child progressed, and one regressed! All told, we can claim some success for mapping instruction in helping children to transfer their understanding of blocks arithmetic into the domain of written arithmetic. However, there was nothing like total transfer of understanding from blocks to writing. One of the most important tasks ahead of us now is accounting for individual differences in learning from mapping instruction.

The results for written calculation errors were more surprising. Neither group improved reliably between the pretest and the second posttest, although there was a temporary improvement for many children at the time of the immediate posttest. For most children, old bugs remained or reappeared at the second posttest, or new ones were invented. We expected such an outcome for the Prohibition group because our theory, as we began the experiment, was that only a command of the semantic principles of subtraction would successfully block the use or construction of buggy procedures that violate those principles. But we did expect improvement in understanding to produce improvement in calculation performance for the Mapping subjects. It did not reliably do so.

(a) E: I want to show you some problems that someone else did and see if you can tell whether this person did it correctly or not. This is the problem:

  5̸0̸6
− 178
  332

See if you can check that, and check all the steps and make sure it was done correctly. If you see something wrong, tell me what's wrong.

S: She left it a ten--kept it a ten.

E: What should she have done?

S: Made it a 9.

E: Why is that?

S: To take 90 tens from here (hundreds) and then the other 10 would go there (the ones).

E: How many do you take from here (hundreds) altogether?

S: A hundred.

(b) E: OK, so how do you write that?

S: You put 10 there (13) and 9 there (in tens) which is 90 and 9 there (hundreds), which is 900. (Writes: [illegible])

E: OK. So where are the 10 hundreds in the writing?

S: 100 is right here (points to top digits in the ones column and tens columns) and 900 is right here (points to the hundreds column).

Figure 5. Two extracts from a child's explanations of written borrowing. (From Resnick, 1983).

A closer look at the data for the Mapping groups suggests two possible interpretations. One is that calculation performance improves only when understanding improves to a very high level. Partial understanding—especially understanding that does not include the compensation principle—may not be adequate to block buggy performance. This interpretation is consonant with our data. The two children, Jan and Dav, who reached the highest level of understanding also eliminated all bugs in the delayed posttest. The other children who improved (Am, Em, and Den), but not to the top levels, showed little or no change in the number or kinds of bugs in their calculation. We cannot reach a firm conclusion on the basis of the present data, however, because the number of subjects is too small.

The second possible interpretation is that principles that can be used by a child to construct

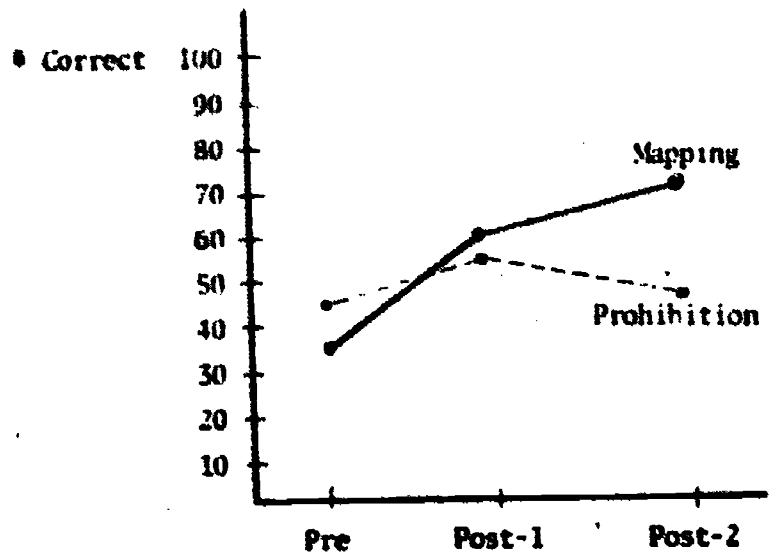

Figure 6. Correct responses by Mapping and Prohibition groups on value questions, showing change in understanding of value principle. (From Omanson, 1982.)

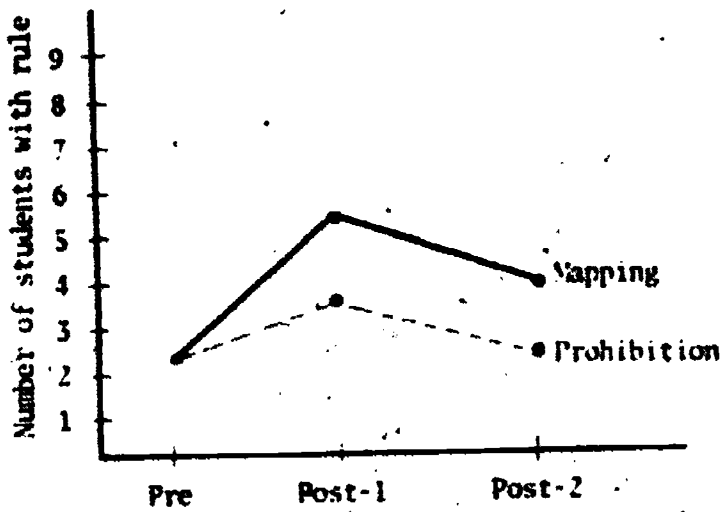

Figure 7. Number of students from Mapping and Prohibition groups showing understanding of compensation principle. (From Omanson, 1982.)

explanations and justifications of a procedure are not necessarily automatically applied to performance of that procedure. This explanation would constitute a strong challenge to our initial predictions. Clearly we need replication and some extension of this research in order to decide between the two interpretations. This work in now under way.

Conclusions and Some Questions

Although we cannot draw firm conclusions until the results of our further research are in hand, it is nevertheless useful to reflect now on the implications of our present findings for the general questions about arithmetic learning raised in the course of this paper.

The central question we have been considering is the relationship between understanding of basic principles of number and numeration and performance of arithmetic procedures. It is clear that buggy performances involve violations of basic principles, and thus it seemed reasonable to suppose that if children acquired the principles they would be less likely to engage in buggy performances. Implicit in this prediction was the assumption that children would call upon all relevant resources whenever they performed arithmetic procedures--i.e., that if they knew principles, they would use them in performance. If our replication research shows that even high levels of understanding do not reliably suppress buggy calculation performances, this assumption will have to be abandoned.

A careful analysis of Brown and VanLehn's repair theory suggests why it might be so difficult for newly acquired principles to affect calculation performance in children. Repair theory has been formalized as a running computer program that invents most of the same bugs that children regularly invent but does not invent other logically possible bugs. Because its performance is so similar to children's, it is reasonable to take the program as a model of what children do when they invent buggy algorithms. We can then use the known details of the program's functioning to reason about what might be happening when children learn new mathematical principles but fail to apply them to their computational algorithms.

As discussed earlier, the repair theory program produces bugs by generating repairs and checking them against critics. All of the critics in the present repair program are syntactic in nature; they do not reflect the basic principles of subtraction discussed in this paper. Suppose the program were to acquire a set of semantic critics that did reflect the principles. Would that effectively screen out all the buggy repairs? At first blush, it seems reasonable to believe it would.

Careful consideration of the program, however, shows that it is not possible to simply add in the new semantic critics without making much more fundamental changes in the program (VanLehn, personal communication). This is because the present program treats subtraction as a system of operations on symbols rather than as a system of operations on quantities. It "knows" what symbolic marks need to be made to do the incrementing and decrementing involved in borrowing. But there is nothing in its knowledge base to represent the fact that when it puts an increment mark in the units column it is adding 10, or that when it decrements the number in the hundreds column by one it is really subtracting 100. This being the case, a critic that checks for whether the total quantity has been maintained has nothing to refer to; the program has no representation at all of the total quantity or of the subquantities being transferred in the course of a borrow operation. To incorporate critics that refer to principles of quantity, such as the ones discussed in this paper, it would be necessary to fundamentally change the entire way in which the program represents subtraction.

In much the same way, absence of a quantity representation when performing written subtraction may be the source of children's difficulty in incorporating the principles. If, when they are doing calcula-

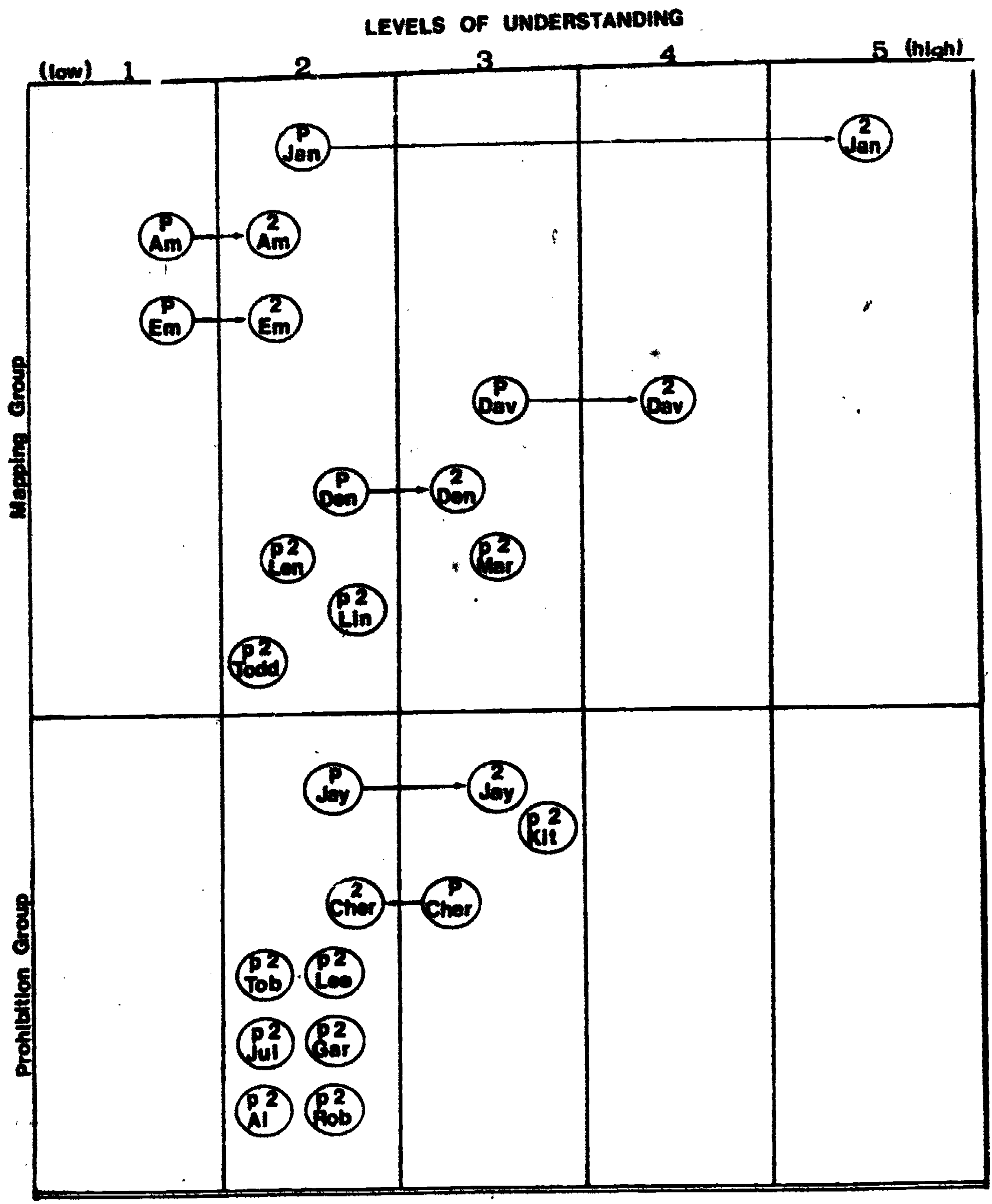

Figure 8. Individual students' changes in understanding of written subtraction. (From Omanson, 1982.)

tion, children do not represent the problem to themselves as involving quantities but only as digits to be manipulated, then there is no simple way for them to apply their newly learned principles. They must first interrupt their normal performance to rerepresent the problem for themselves as one involving operations on quantities. This, however, means giving up all the efficiency of an "automated" skill and requires paying attention to every step. Especially for children who have been doing subtraction for several years (albeit with a certain percentage of errors), rerepresenting is something they are likely to do only when some special constraint, such as an experimenter monitoring each step, is imposed.

What would this interpretation imply for instruction? First, we would have to conclude that simply explaining and demonstrating the principles of place value arithmetic to children would not have much of an effect on their calculation performance. Even improving children's understanding to the point where they could construct explanations themselves could not be counted on to eliminate buggy calculation rules once children had adopted a more or less automatic procedure.

One possibility—now being explored by my colleague, Audrey Champagne—is that early focus on the principles of representing total quantities in decimal notation and making appropriate exchanges among the parts of a total quantity would prevent buggy rules from ever becoming automated. This would imply much more sustained attention than is now typical in instruction both to the basic principles of the decimal number system and to the ways in which these principles are incorporated into the written numeration system. Most arithmetic textbooks do attempt to explain and demonstrate the rationale for carrying and borrowing, often using pictorial representations similar to Dienes blocks. However, instructional attention passes quickly to efficient calculation, thereby probably encouraging automation of calculation rules that are not well linked to the principles. If Champagne's preliminary results are an indication, it will require much more extensive attention to the principles than is now common, and much more explicit linkage of principles to written notation, to achieve the desired results.

For children such as those in our experiment, who have already acquired buggy and fairly automatic calculation routines, our findings suggest that learning principles alone probably will not be enough to correct calculation errors. Instruction will probably have to focus directly on the calculation performance. This might take the form of deliberately invoking, and maintaining for as long as necessary, a reflective attitude toward calculation that would include thinking about how the basic principles apply to each step of a calculation procedure. Such instruction would be in the spirit of current research on teaching self-monitoring skills for complex tasks such as reading comprehension (see Palincsar & Brown, 1984). Our mapping instruction did not systematically do this. At the very end of the mapping sequence, children were asked to perform only the written subtraction but to think about the steps "as if there were a way to write down what one did with the blocks." The protocols of the instructional sessions suggest that only some children did reflect in this way. Furthermore, thinking about writing as a record of blocks action does not automatically ensure that one is thinking about the principles that underlie both blocks and writing. Our instruction thus did not fully test the possibilities for bringing principles to bear on calculation. Even with a more powerful version of principle-based instruction, however, it seems likely that direct attention to re-automating a new and correct calculation procedure will also be needed.

What roles can we envisage for systematic diagnosis of errors in the kind of instruction just outlined? I can think of two. First, the process of reflecting on performance and on the relationship of procedures to underlying principles may be significantly enhanced by asking children to identify buggy procedures and to explain how these procedures violate the principles of arithmetic. For this approach to work, it may not prove necessary for children to reflect on their own incorrect procedures, but it would certainly be important for them to reflect on typical errors. These are the errors that arise from the same kinds of reasoning the children are likely to engage in themselves. Thus, for each major procedural domain in arithmetic, we would need to know what the most likely buggy procedures are.

Instruction focused more directly on automated procedural skill can also benefit from knowledge of buggy algorithms. First, for any given child, it is probably important to know whether errors in calculation are due to systematically faulty rules or to careless errors ("slips" rather than "bugs"). Children who routinely make calculation slips probably need a different kind of arithmetic practice than if the elementary operations are correct but the processes are improperly put together. Second, it may prove useful to tailor practice to specific kinds of buggy rules, either by choosing particular examples that are matched to a child's bugs or by giving special attention to the parts of a procedure that evoke those bugs. In either case, diagnostic tests capable of detecting specific buggy algorithms will play an important role in instruction.

The current evidence, then, confirms the importance of error analysis research both as a basis for studies of arithmetic learning processes and as a potential instructional tool. Our research also shows, however, that systematic errors probably arise from a basic failure to mentally represent arithmetic procedures in terms of operations on quantities within a principled number system, rather than as operations on symbols that obey largely syntactic rules. Instruction probably needs to focus more explicitly and for a much longer time than it now does on procedures as reflections of fundamental principles. Error analysis can provide a framework for this kind of instruction, but it is not clear that detailed diagnosis of individual children's buggy rules is required. Furthermore, it seems likely that understanding alone does not reliably produce correct computational procedures. Direct instructional attention to the problem of de-automating incorrect procedures and replacing them with correct and automatic procedures will also be needed—at least if computational skill is accepted as a goal of mathematics instruction. As we attend to this problem, some of the recent work on self-monitoring and meta-cognitive skills may prove an important source of theoretical and practical ideas.

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